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Associative systems of left quotients ☆

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Abstract

In this paper we construct the maximal left quotient system of every pair of modules in some generalized matrix rings. These rings can be seen as 3-graded algebras or as superalgebras. We show the relation among the three different notions of left quotients.

Keywords: Systems of quotients; Associative pair; Associative triple system; Graded algebra; Superalgebra

Introduction

Over the years, various notions of quotient structures, such as the field of fractions of an integral domain, or the classical ring of quotients of a ring satisfying the Ore condition, have played an important role in Modern Algebra.

The notion of left quotient ring, introduced by Utumi in 1956 (see [18]), is more general than the ones above, as well as those of maximal symmetric ring of quotients (introduced by Lanning in [10]), and Martindale's symmetric ring of quotients (see [13]). The existence of a maximal left quotient ring for every not necessarily unital ring without total right zero

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divisors (proved by Utumi) provides a common framework where to settle the different types of rings of quotients.

In the associative context, not only rings (or algebras) can be considered. The study of systems of quotients in structures such as associative pairs or associative triple systems could be crucial in order to shed some light on the structure theory of Jordan systems (algebras, pairs or triples) and of Lie algebras, via the theory of quotients. This approach is having a great development. See the works [2,6,14,16,17] on the theory of quotients of Jordan systems and Lie algebras.

Associative pairs play a fundamental role in the new approach (see [5]) to Zelmanov's classification of strongly prime Jordan pairs, and have been already used by Loos in the classification of the nondegenerate Jordan pairs of finite capacity [11].

In our work we will give a pair and triple system version of the maximal left quotient ring. A first attempt was made in [7], where the authors found the maximal left quotient pair of a right faithful associative pair in the left faithful or left nonsingular cases.

If $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ is a superalgebra then $\mathcal{A}_1$ can be seen as a triple system, while if $\mathcal{B} = \mathcal{B}_{-1} \oplus \mathcal{B}_0 \oplus \mathcal{B}_1$ is a 3-graded algebra, then $(\mathcal{B}_{-1}, \mathcal{B}_1)$ has a structure of associative pair. And conversely, every associative pair $A = (A^+, A^-)$ (or triple system T) can be embedded in an algebra $\mathcal{E}$ with an idempotent e such that (A^+, A^-) ((T, T) in the triple system case) can be identified with $(e\mathcal{E}(1-e), (1-e)\mathcal{E}e)$. This algebra $\mathcal{E}$ has a supergrading $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1$, where $\mathcal{E}_0 = e\mathcal{E}e \oplus (1-e)\mathcal{E}(1-e)$, $\mathcal{E}_1 = e\mathcal{E}(1-e) \oplus (1-e)\mathcal{E}e$, and a 3-grading $\mathcal{E} = \mathcal{E}_{-1} \oplus \mathcal{E}_0 \oplus \mathcal{E}_1$, for $\mathcal{E}_{-1} = (1-e)\mathcal{E}e$, $\mathcal{E}_0 = e\mathcal{E}e \oplus (1-e)\mathcal{E}(1-e)$ and $\mathcal{E}_1 = e\mathcal{E}(1-e)$. So that it seems to be quite natural to relate the study of graded left quotient algebras of a graded algebra (see [4] for a construction of the gr-maximal left quotient algebra of a not necessarily unital gr-algebra) to that of left quotient systems of an associative triple system or pair.

On the other hand, in some cases (for example, when $\mathcal{E}$ is simple) every standard envelope gives rise to a surjective Morita context for not necessarily unital rings, and conversely, every pair of bimodules of a Morita context has a natural structure of associative pair. Hence, in particular, our work can be considered as an approach to the study of maximal rings of quotients of Morita contexts for not necessarily unital rings.

The paper is divided into five sections. After a preparatory section where right faithfulness is studied, we introduce in Section 1 the notion of subpair of a 3-graded algebra. Proposition 1.5 provides a useful tool to compute the standard envelope of any right faithful associative pair (Corollary 1.6). In Section 2 we study the supersingular ideal of a not necessarily unital superalgebra $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ and relate it to the singular ideals of $\mathcal{A}_0$ (as an algebra) and of $\mathcal{A}_1$ (as an associative triple system). In the following section we introduce the notion of weak right faithful superalgebra in an oversuperalgebra and relate left quotient algebras, left quotient triple systems and left quotient superalgebras: Suppose that $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$, with $\mathcal{A}_0 = \mathcal{A}_1\mathcal{A}_1$, is a weak right faithful superalgebra in an oversuperalgebra $\mathcal{B}$. Then $\mathcal{B}$ is a gr-left quotient algebra of $\mathcal{A}$ if and only if $\mathcal{B}_1$ is a left quotient triple system of $\mathcal{A}_1$ and $\mathcal{B}_0$ is a left quotient algebra of $\mathcal{A}_0$. Weak right faithfulness is just the condition needed to have a result allowing to go back and forth between left quotient algebras and left quotient systems to left quotient superalgebras. Examples of right faithful subsuperalgebras in overalgebras are every left quotient algebra of a faithful, or left nonsingular superalgebra. As a consequence of the previous results, in Section 4 we construct the maximal left quotient pair of a right faithful associative pair. This maximal left quotient pair is given in terms of the maximal left quotient algebra of its envelope, which coincides with the graded maximal left quotient algebra of this envelope, considered as a 3-graded algebra, or as a superalgebra. We show that this construction, which was made in [7] for some particular cases, cannot be improved. In Section 5 we proceed analogously in the triple system case.

0. Preliminaries

0.1. We will deal with associative systems (algebras, pairs, and triple systems) over an arbitrary (unital commutative associative) ring of scalars Φ. Recall that an **associative pair** over Φ is a pair of Φ-modules (A^+, A^-) together with a pair of trilinear maps

$$\langle\, ,\, ,\,\rangle^\sigma : A^\sigma \times A^{-\sigma} \times A^\sigma \to A^\sigma, \quad \sigma = \pm,$$

satisfying

$$\big\langle \langle x, y, z\rangle^\sigma, u, v\big\rangle^\sigma = \big\langle x, \langle y, z, u\rangle^{-\sigma}, v\big\rangle^\sigma = \big\langle x, y, \langle z, u, v\rangle^\sigma\big\rangle^\sigma$$

for any $x, z, v \in A^\sigma$, $y, u \in A^{-\sigma}$, $\sigma = \pm$.

Similarly, an **associative triple system** A over Φ is a Φ-module equipped with a trilinear map

$$\langle\, ,\, ,\,\rangle : A \times A \times A \to A,$$

satisfying

$$\big\langle \langle x, y, z\rangle, u, v\big\rangle = \big\langle x, \langle y, z, u\rangle, v\big\rangle = \big\langle x, y, \langle z, u, v\rangle\big\rangle,$$

for any $x, y, z, u, v \in A$. We can also consider the **opposite** associative pair $A^{\mathrm{op}} = (A^+, A^-)$ obtained by reversing the products of A ($\langle x, y, z\rangle^\sigma_{\mathrm{op}} = \langle z, y, x\rangle^\sigma$).

As for pairs, one can consider the opposite triple system A^{op} of A.

Due to associativity, there is no risk of ambiguity when deleting the brackets $\langle\ \rangle$, thus, the products above will be usually denoted by juxtaposition, just like in the associative algebra case.

0.2. An associative algebra A gives rise to the associative triple system A_T by simply restricting to odd length products. By doubling any associative triple system A one obtains the **double associative pair** $V(A) = (A, A)$ with obvious products. From an associative pair $A = (A^+, A^-)$ one can get a (**polarized**) associative triple system $T(A) = A^+ \oplus A^-$ by defining $(x^+ \oplus x^-)(y^+ \oplus y^-)(z^+ \oplus z^-) = x^+y^-z^+ \oplus x^-y^+z^-$.

0.3. Given an associative pair $A = (A^+, A^-)$, and elements $x, z \in A^\sigma$, $y \in A^{-\sigma}$, $\sigma = \pm$, recall that **left**, **middle** and **right multiplications** are defined by

$$\lambda(x,y)z=\mu(x,z)y=\rho(y,z)x=xyz. \tag{1}$$

From 0.1 and (1), for any $x,u\in A^\sigma,\ y,v\in A^{-\sigma}$,

$$\lambda(x,y)\lambda(u,v)=\lambda(xyu,v)=\lambda(x,yuv), \tag{2}$$

and similarly

$$\rho(u,v)\rho(x,y)=\rho(x,yuv)=\rho(xyu,v). \tag{3}$$

As a consequence of (2) and (3), it is clear that the linear span of all operators $T:A^\sigma\to A^\sigma$ of the form $T=\lambda(x,y)$, for $(x,y)\in A^\sigma\times A^{-\sigma}$, or $T=Id_{A^\sigma}$ is a unital associative algebra; it will be denoted by $\Lambda(A^\sigma,A^{-\sigma})$. Clearly A^σ is a left $\Lambda(A^\sigma,A^{-\sigma})$-module. Similarly, we define $\Pi(A^{-\sigma},A^\sigma)$ as the linear span of all the right multiplications and the identity on A^σ, so that A^σ becomes a left $\Pi(A^{-\sigma},A^\sigma)$-module.

0.4. The well-known notions of left and right ideals of an associative algebra have the following analogues for pairs and triple systems: Given an associative pair A, we define the **left ideals** $L\subset A^\sigma$ of A as the $\Lambda(A^\sigma,A^{-\sigma})$-submodules of A^σ, and the **right ideals** $R\subset A^\sigma$ as the $\Pi(A^{-\sigma},A^\sigma)$-submodules. A **two-sided ideal** $B\subset A^\sigma$ is both a left and a right ideal. An **ideal** $I=(I^+,I^-)$ of A is a pair of two-sided ideals of A such that $A^\sigma I^{-\sigma}A^\sigma\subseteq I^\sigma$, $\sigma=\pm$.

For an associative triple system A, the left and right ideals of A are simply those of the pair $V(A)$, while an ideal I of A is a left and right ideal also satisfying $AIA\subseteq I$, i.e., a Φ-submodule I of A such that $V(I)$ is an ideal of $V(A)$.

Notice that, if I is a left or right ideal of an associative algebra A, then it is a left or right ideal, respectively, of the associative triple system A_T. Similarly, an ideal of A is always an ideal of A_T.

0.5. Recall that given a group G (not necessarily abelian), an algebra A is said to be G-**graded** if $A=\bigoplus_{\sigma\in G}A_\sigma$, where A_σ is a Φ-subspace of A and $A_\sigma A_\tau\subseteq A_{\sigma\tau}$ for every $\sigma,\tau\in G$. We will say that A is **strongly graded** if $A_\sigma A_\tau=A_{\sigma\tau}$. In the sequel, we will use “graded” instead of “G-graded” when the group is understood. As usual, by the prefix “gr-” we mean “graded-”.

We will say that an algebra is **3-graded** if $G=\mathbb{Z}$ and $A=A_{-1}\oplus A_0\oplus A_1$. When $\mathbb{Z}=\mathbb{Z}/2\mathbb{Z}$ we will speak about a **superalgebra**. We will use [15] as a standard reference for graded algebras and modules.

In a graded algebra $A=\bigoplus_{\sigma\in G}A_\sigma$, each element of A_σ is called an **homogeneous element**.

The neutral element of G will be denoted by e.

A **graded left ideal** I of a G-graded algebra A is a left ideal such that if $x=\sum_{\sigma\in G}x_\sigma\in I$, then $x_\sigma\in I$ for every $\sigma\in G$.

0.6. Let $\mathcal{A}$ be an associative algebra. An element $a\in\mathcal{A}$ is said to be a **total right zero divisor** if $\mathcal{A}a=0$. Similarly, an element $a\in A^\sigma$ of an associative pair is called a **total**

right zero divisor if $A^\sigma A^{-\sigma} a = 0$. A pair or an algebra not having nonzero total right zero divisors will be called **right faithful**.

0.7. An homogeneous element x_σ of a gr-algebra $A = \bigoplus_{\sigma \in G} A_\sigma$ is called an **homogeneous total right zero divisor** if it is nonzero and a total right zero divisor, that is, $Ax_\sigma = 0$. By [4, (1.10)], a gr-algebra is right faithful if and only if it has no homogeneous total right zero divisors.

0.8. Definitions. A **total right zero divisor** in an associative triple system S is a nonzero element $s \in S$ such that $SSs = 0$, equivalently, s is a total right zero divisor in the associative pair $V(S)$. An associative triple system without total right zero divisors will be called **right faithful**.

Given a superalgebra $A = A_0 \oplus A_1$, the odd part has a structure of associative triple system, while the even part is an algebra. Now, we show the relation of faithfulness among the three structures.

0.9. Lemma. *Let $A = A_0 \oplus A_1$ be a superalgebra. If A_0 and A_1 are right faithful, then A is right faithful too. The converse is true if $A_0 = A_1 A_1$.*

Proof. Suppose $A_0 = A_1 A_1$ and that A has no total right zero divisors. By [4, (3.1)], A_0 has no total right zero divisors. If $a_1 \in A_1$ satisfies $A_1 A_1 a_1 = 0$, then $AAa_1 = (A_1 A_1 + A_1)(A_1 A_1 + A_1)a_1 \subseteq (A_1 A_1 A_1 A_1 + A_1 A_1 A_1 + A_1 A_1)a_1 = 0$. Apply twice that A is right faithful to have $a_1 = 0$.

The converse is straightforward. $\square$

0.10. Remark. The condition $A_0 = A_1 A_1$ in the previous lemma cannot be removed. If a superalgebra $A = A_0 \oplus A_1$ is right faithful, then A_0 is a right faithful algebra [4, (3.1)], but A_1 is not necessarily a right faithful associative triple system: Let F be an arbitrary field and consider the F-algebra $A = F[x]/\langle x^3 \rangle$, where $\langle x^3 \rangle$ denotes the ideal generated by x^3 inside $F[x]$. For an element $u \in F[x]$, let $\bar{u}$ stand for the class of u in A. Then the superalgebra $A = A_0 \oplus A_1$, with A_0 the subalgebra of A generated by $\{\bar{1}, \bar{x}^2\}$ and A_1 the vector subspace of A generated by $\{\bar{x}\}$, is a right faithful algebra but A_1 is not a right faithful associative triple system because $A_1 A_1 \bar{x} = \bar{0}$ while $\bar{x} \neq 0$. Notice that $A_1 A_1 \neq A_0$ because $\bar{1} \notin (F\{\bar{x}\})^2$.

0.11. Remark. Although we always work with systems of left quotients, the results in this paper have their right-side analogues, with obvious changes in the definitions, just reversing products in the proofs or applying the left-side results to the opposite systems.

1. Algebra envelopes of associative pairs

In this section we give a method to determine the standard envelope of an associative pair without total right zero divisors by means of any graded algebra containing the pair in a suitable way and generated by it.

1.1. Associative pairs are really "abstract off-diagonal Peirce spaces" of associative algebras. Let $\mathcal{E}$ be a unital associative algebra. Consider the Peirce decomposition $\mathcal{E} = \mathcal{E}_{11} \oplus \mathcal{E}_{12} \oplus \mathcal{E}_{21} \oplus \mathcal{E}_{22}$ of $\mathcal{E}$ with respect to an idempotent $e \in \mathcal{E}$, i.e.,

$$\mathcal{E}_{11} = e\mathcal{E}e, \qquad \mathcal{E}_{12} = e\mathcal{E}(1-e), \qquad \mathcal{E}_{21} = (1-e)\mathcal{E}e \quad \text{and} \quad \mathcal{E}_{22} = (1-e)\mathcal{E}(1-e).$$

From the Peirce multiplication rules, $(\mathcal{E}_{12}, \mathcal{E}_{21})$ is a subpair of $V(\mathcal{E})$. Conversely, every associative pair $A = (A^+, A^-)$ can be obtained in this way (see [12, (2.3)]): Let $\mathcal{C}$ be the Φ-submodule of $\mathcal{B} = End_\Phi(A^+) \times End_\Phi(A^-)^{\mathrm{op}}$ spanned by $e_1 = (Id_{A^+}, Id_{A^-})$ and all $(\lambda(x,y), \rho(x,y))$, and similarly, let $\mathcal{D}$ be the submodule of $\mathcal{B}^{\mathrm{op}}$ spanned by $e_2 = (Id_{A^+}, Id_{A^-})$ and all $(\rho(y,x), \lambda(y,x))$ where $(x,y) \in A^+ \times A^-$. By 0.3, these Φ-linear spans are really subalgebras. Clearly, A^+ is an $(\mathcal{C}, \mathcal{D})$-bimodule if we set

$$cx = c^+(x), \qquad xd = d^+(x)$$

for $x \in A^+$ and $c = (c^+, c^-) \in \mathcal{C}$, $d = (d^+, d^-) \in \mathcal{D}$. Similarly, A^- is a $(\mathcal{D}, \mathcal{C})$-bimodule. Now we define bilinear maps on $A^\pm \times A^\mp$ with values in $\mathcal{C}$, respectively, $\mathcal{D}$, by

$$xy = \big(\lambda(x,y), \rho(x,y)\big), \qquad yx = \big(\rho(y,x), \lambda(y,x)\big).$$

Then it is easy to check that $(\mathcal{C}, A^+, A^-, \mathcal{D})$ is a Morita context which gives rise to a unital associative algebra $\mathcal{E}$ (cf. [12, (2.3)]). If we set $e = e_1$, then the pair $A = (A^+, A^-)$ is isomorphic to the associative pair $(\mathcal{E}_{12}, \mathcal{E}_{21})$. Moreover, $\mathcal{E}_{11}$ (respectively, $\mathcal{E}_{22}$) is spanned by e and all products $x_{12}y_{21}$ (respectively, $1-e$ and all products $y_{21}x_{12}$) for $x_{12} \in \mathcal{E}_{12}$, $y_{21} \in \mathcal{E}_{21}$, and has the property that

$$x_{11}\mathcal{E}_{12} = \mathcal{E}_{21}x_{11} = 0 \Rightarrow x_{11} = 0, \qquad x_{22}\mathcal{E}_{21} = \mathcal{E}_{12}x_{22} = 0 \Rightarrow x_{22} = 0. \tag{1}$$

1.2. Let $\mathcal{A}$ be the subalgebra of $\mathcal{E}$ generated by $\mathcal{E}_{12} \cup \mathcal{E}_{21}$, i.e., $\mathcal{A} = \mathcal{E}_{12} \oplus \mathcal{E}_{12}\mathcal{E}_{21} \oplus \mathcal{E}_{21}\mathcal{E}_{12} \oplus \mathcal{E}_{21}$. It is immediate that $\mathcal{A}$ is an ideal of $\mathcal{E}$. We will call $\mathcal{A}$ the **standard envelope of** the associative pair A, and will write $\tau = (\tau^+, \tau^-)$ for the natural inclusion $\tau^\sigma : A^\sigma \to \mathcal{A}$ of A into $\mathcal{A}$. When it is necessary to emphasize the existence of the idempotent e (see 1.1) we will write $(\mathcal{A}, e)$ instead of merely $\mathcal{A}$. The pair $(\mathcal{E}, e)$ is called the **standard imbedding of** A.

1.3. Definition. Let A be an associative pair, $\mathcal{A} = \mathcal{A}_{-1} \oplus \mathcal{A}_0 \oplus \mathcal{A}_1$ be a 3-graded associative algebra, and $\varphi = (\varphi^+, \varphi^-)$, where $\varphi^\sigma : A^\sigma \to \mathcal{A}$ is an injective Φ-linear map, $\sigma = \pm$. We will say that A is a **subpair of** $(\mathcal{A}, \varphi)$ if

(i) $\varphi^+(A^+) \subseteq \mathcal{A}_1$ and $\varphi^-(A^-) \subseteq \mathcal{A}_{-1}$ and
(ii) $\varphi : A \to V(\mathcal{A})$ is a pair homomorphism (hence monomorphism).

When A is a subpair of $(\mathcal{A}, \varphi)$ then $\varphi^+(A^+) + \varphi^+(A^+)\varphi^-(A^-) + \varphi^-(A^-)\varphi^+(A^+) + \varphi^-(A^-)$ is a subalgebra of $\mathcal{A}$. If it coincides with $\mathcal{A}$ (i.e., $\varphi^+(A^+) \cup \varphi^-(A^-)$ generates $\mathcal{A}$ as an algebra), the pair $(\mathcal{A}, \varphi)$ is called a **graded envelope of** A (**gr-envelope**, for short).

In this case, and equivalently,

(iii) $\mathcal{A}_1 = \varphi^+(A^+)$, $\mathcal{A}_{-1} = \varphi^-(A^-)$, $\mathcal{A}_0 = \varphi^+(A^+)\varphi^-(A^-) + \varphi^-(A^-)\varphi^+(A^+)$.

1.4. Remark. Notice that for an associative pair A the standard envelope $(\mathcal{A}, \tau)$ of A, which can be seen as a 3-graded algebra by considering $\mathcal{A}_1 = \mathcal{E}_{12}$, $\mathcal{A}_0 = \mathcal{E}_{12}\mathcal{E}_{21} \oplus \mathcal{E}_{21}\mathcal{E}_{12}$ and $\mathcal{A}_{-1} = \mathcal{E}_{21}$, is a gr-envelope of A in the sense above.

If an associative pair A is a subpair of a $(\mathcal{A}, \varphi)$, with $\mathcal{A}$ a 3-graded algebra, in the sense of Definition 1.3, then A is a subpair of $(\mathcal{A}, \varphi)$ in the sense of [1, (1.3)] because $\varphi^+(A^+) \cap \varphi^-(A^-) \subseteq \mathcal{A}_1 \cap \mathcal{A}_{-1} = 0$.

An envelope $(\mathcal{A}, \varphi)$ of A will be called **tight** if every nonzero ideal of $\mathcal{A}$ hits A ($I \cap (\varphi^+(A^+) \cup \varphi^-(A^-)) \neq 0$ for every nonzero ideal I of $\mathcal{A}$). We will say that $(\mathcal{A}, \varphi)$ and $(\tilde{\mathcal{A}}, \tilde{\varphi})$ are **isomorphic envelopes** of A if there exists an algebra isomorphism $\psi : \mathcal{A} \to \tilde{\mathcal{A}}$ such that $\psi \circ \varphi^\sigma = \tilde{\varphi}^\sigma$, $\sigma = \pm$.

The proof of the following result follows partially [1, (1.5)]. Notice that it is more general in the sense that we have replaced left and right faithfulness in [1, (1.5)] with right faithfulness by considering gr-envelopes instead of envelopes.

1.5. Proposition. *Let $\mathcal{A} = \mathcal{A}_{-1} \oplus \mathcal{A}_0 \oplus \mathcal{A}_1$ be a 3-graded algebra which is a gr-envelope of a right faithful associative pair A. Then*:

(i) *Every one-sided gr-ideal of $\mathcal{A}$ not hitting $\varphi(A)$ is contained in $\mathcal{A}_0$.*

Define as in [1, (1.5)],

$$\begin{aligned}
\mathcal{I} &= \bigl\{x \in \varphi^+\bigl(A^+\bigr)\varphi^-(A^-) + \varphi^-(A^-)\varphi^+\bigl(A^+\bigr) \mid x\varphi^\sigma\bigl(A^\sigma\bigr) = 0 = \varphi^\sigma\bigl(A^\sigma\bigr)x, \sigma = \pm\bigr\} \\
&= \bigl\{x \in \varphi^+\bigl(A^+\bigr)\varphi^-(A^-) \mid x\varphi^+\bigl(A^+\bigr) = 0 = \varphi^-(A^-)x\bigr\} \\
&\quad + \bigl\{x \in \varphi^-(A^-)\varphi^+\bigl(A^+\bigr) \mid x\varphi^-(A^-) = 0 = \varphi^+\bigl(A^+\bigr)x\bigr\}.
\end{aligned}$$

(ii) *$\mathcal{I} \subseteq \mathcal{A}_0$, it is the biggest gr-ideal of $\mathcal{A}$ not hitting $\varphi(A)$ and it satisfies $\mathcal{I}\mathcal{A}_i = \mathcal{A}_i\mathcal{I} = 0$ for $i = 0, \pm 1$.*

(iii) *Define*

$$\begin{aligned}
\phi^\sigma : A^\sigma &\to \mathcal{A}/\mathcal{I} \\
x^\sigma &\mapsto \overline{\varphi^\sigma(x^\sigma)}
\end{aligned}$$

where $\bar{a}$ denotes the class of the element a of $\mathcal{A}$ inside $\mathcal{A}/\mathcal{I}$, $\sigma = \pm$. Then $(\mathcal{A}/\mathcal{I}, \phi)$ is a gr-envelope of A gr-isomorphic to the standard envelope of A.

Proof. (i) Let $\mathcal{J} = \mathcal{J}_{-1} \oplus \mathcal{J}_0 \oplus \mathcal{J}_1$ be a one-sided gr-ideal of $\mathcal{A}$ not hitting $\varphi(A)$. Since $\mathcal{J}_{\pm 1} \subseteq \mathcal{J} \cap \mathcal{A}_{\pm 1} \subseteq \mathcal{J} \cap \varphi(A) = 0$, $\mathcal{J} \subseteq \mathcal{A}_0$.

(ii) From Definition 1.3, it is clear that $\mathcal{I}$ is an ideal of $\mathcal{A}$ and that both definitions of $\mathcal{I}$ agree. Moreover, by (i) and the definition, $\mathcal{I} \subseteq \mathcal{A}_0$ and $\mathcal{A}_i x = x\mathcal{A}_i = 0$ for any $x \in \mathcal{I}$ and every $i = 0, \pm 1$.

Now, let $\mathcal{J}$ be a gr-ideal of $\mathcal{A}$ not hitting $\varphi(A)$. By (i), $\mathcal{J} \subseteq \mathcal{A}_0$. Take $0 \neq y_0 \in \mathcal{J}$ and write $y_0 = \sum_{i=1}^m \varphi^+(u_i^+)\varphi^-(u_i^-) + \sum_{j=1}^n \varphi^-(v_j^-)\varphi^+(v_j^+)$, with $u_i^\sigma, v_j^\sigma \in A^\sigma$, $\sigma = \pm$. Suppose $(\sum_{i=1}^m \lambda(u_i^+, u_i^-), \sum_{i=1}^m \rho(u_i^+, u_i^-)) \neq 0$. Then, by the proof of [7, (2.6)], there exists $a^- \in A^-$ such that $0 \neq \sum_{i=1}^m a^- u_i^+ u_i^-$. Since φ is an injective Φ-linear map, and by Definition 1.3(ii), $0 \neq \sum_{i=1}^m \varphi^-(a^-)\varphi^+(u_i^+)\varphi^-(u_i^-) = \varphi^-(a^-)y_0 \in \mathcal{I} \cap \varphi^-(A^-) = 0$, a contradiction. Hence

$$\left(\sum_{i=1}^m \lambda\big(u_i^+, u_i^-\big), \sum_{i=1}^m \rho\big(u_i^+, u_i^-\big)\right) = 0.$$

Similarly,

$$\left(\sum_{j=1}^n \lambda\big(v_j^-, v_j^+\big), \sum_{j=1}^n \rho\big(v_j^-, v_j^+\big)\right) = 0.$$

This means that for every $(x^+, x^-) \in A$,

$$\sum_{i=1}^m u_i^+ u_i^- x^+ = 0, \qquad \sum_{i=1}^m x^- u_i^+ u_i^- = 0, \qquad \sum_{j=1}^n x^+ v_j^- v_j^+ = 0 \quad \text{and} \quad \sum_{j=1}^n v_j^- v_j^+ x^- = 0.$$

Apply φ and Definition 1.3(ii) to these identities to obtain:

$$0 = \sum_{i=1}^m \varphi^+\big(u_i^+\big)\varphi^-(u_i^-)\varphi^+\big(x^+\big) = \left(\sum_{i=1}^m \varphi^+\big(u_i^+\big)\varphi^-(u_i^-)\right)\varphi^+\big(x^+\big) = y_0\varphi^+\big(x^+\big),$$

$$0 = \sum_{i=1}^m \varphi^-(x^-)\varphi^+\big(u_i^+\big)\varphi^-(u_i^-) = \varphi^-(x^-)\sum_{i=1}^m \varphi^+\big(u_i^+\big)\varphi^-(u_i^-) = \varphi^-(x^-)y_0,$$

$$0 = \sum_{j=1}^n \varphi^+\big(x^+\big)\varphi^-(v_j^-)\varphi^+\big(v_j^+\big) = \varphi^+\big(x^+\big)\sum_{j=1}^n \varphi^-(v_j^-)\varphi^+\big(v_j^+\big) = \varphi^+\big(x^+\big)y_0,$$

$$0 = \sum_{j=1}^n \varphi^-(v_j^-)\varphi^+\big(v_j^+\big)\varphi^-(x^-) = \left(\sum_{j=1}^n \varphi^-(v_j^-)\varphi^+\big(v_j^+\big)\right)\varphi^-(x^-) = y_0\varphi^-(x^-).$$

This shows $y_0 \in \mathcal{I}$.

(iii) To see the injectivity of the Φ-linear map ϕ^σ, for $\sigma = \pm$, consider $x^\sigma \in A^\sigma$ such that $\overline{\varphi^\sigma(x^\sigma)} = \bar{0}$. This means $\varphi^\sigma(x^\sigma) \in \varphi^\sigma(A^\sigma) \cap \mathcal{I} = 0$.

It is straightforward that $(\mathcal{A}/\mathcal{I}, \phi)$ satisfies Definition 1.3(i)–(iii). This means that it is a gr-envelope of A.

Let $(\tilde{\mathcal{A}}, \tau)$ be the standard envelope of A. We can define a linear map $\psi : \mathcal{A} \to \tilde{\mathcal{A}}$ given by

$$\psi\Bigl(\varphi^+\bigl(x^+\bigr) \oplus \Bigl(\sum_i \varphi^+\bigl(y_i^+\bigr)\varphi^-(y_i^-) + \sum_j \varphi^-(z_j^-)\varphi^+\bigl(z_j^+\bigr)\Bigr) \oplus \varphi^-(u^-)\Bigr)$$
$$= \tau^+\bigl(x^+\bigr) \oplus \sum_i \tau^+\bigl(y_i^+\bigr)\tau^-(y_i^-) \oplus \sum_j \tau^-(z_j^-)\tau^+\bigl(z_j^+\bigr) \oplus \tau^-(u^-),$$

for any $x^+, y_i^+, z_j^+ \in A^+$, $y_i^-, z_j^-, u^- \in A^-$. Indeed, if

$$a = \varphi^+\bigl(x^+\bigr) \oplus \Bigl(\sum_i \varphi^+\bigl(y_i^+\bigr)\varphi^-(y_i^-) + \sum_j \varphi^-(z_j^-)\varphi^+\bigl(z_j^+\bigr)\Bigr) \oplus \varphi^-(u^-) = 0,$$

then $0 = \varphi^+(x^+) = \varphi^-(u^-)$ and by the injectivity of φ, $x^+ = 0$ and $u^- = 0$. Hence $\tau^+(x^+) = 0$ and $\tau^-(u^-) = 0$.

Moreover, $\sum_i \varphi^+(y_i^+)\varphi^-(y_i^-) + \sum_j \varphi^-(z_j^-)\varphi^+(z_j^+) = 0$ implies, if we multiply by $\varphi^-(a^-) \in \varphi^-(A^-)$,

$$0 = \varphi^-(a^-)\sum_i \varphi^+\bigl(y_i^+\bigr)\varphi^-(y_i^-) = \sum_i \varphi^-(a^-)\varphi^+\bigl(y_i^+\bigr)\varphi^-(y_i^-) = \varphi^-\Bigl(\sum_i a^- y_i^+ y_i^-\Bigr).$$

By the injectivity of φ, $0 = \sum_i a^- y_i^+ y_i^-$ and thus

$$0 = \tau^-\Bigl(\sum_i a^- y_i^+ y_i^-\Bigr) = \sum_i \tau^-(a^-)\tau^+\bigl(y_i^+\bigr)\tau^-(y_i^-) = \tau^-(a^-)\sum_i \tau^+\bigl(y_i^+\bigr)\tau^-(y_i^-);$$

similarly, for any $\varphi^+(a^+) \in \varphi^+(A^+)$, $(\sum_i \tau^+(y_i^+)\tau^-(y_i^-))\tau^+(a^+) = 0$, which implies $\sum_i \tau^+(y_i^+)\tau^-(y_i^-) = 0$ by 1.2 and 1.1 (1); in a similar way, $\sum_j \tau^-(z_j^-)\tau^+(z_j^+) = 0$, and we get that ψ is well defined.

It is clear that ψ is a surjective algebra homomorphism of graded algebras satisfying $\psi \circ \varphi^\sigma = \tau^\sigma$, $\sigma = \pm$. By the very definition of ψ, an element a as above lies in $\operatorname{Ker}\psi$ if and only if $a = \sum_i \varphi^+(y_i^+)\varphi^-(y_i^-) + \sum_j \varphi^-(z_j^-)\varphi^+(z_j^+)$ with $\sum_i \tau^+(y_i^+)\tau^-(y_i^-) \oplus \sum_j \tau^-(z_j^-)\tau^+(z_j^+) = 0$, which is shown to be equivalent to $a\varphi^\sigma(A^\sigma) = \varphi^\sigma(A^\sigma)a = 0$, $\sigma = \pm$, using 1.1 (1). Thus $\operatorname{Ker}\psi = \mathcal{I}$, and we can define $\tilde{\psi} : \mathcal{A}/\mathcal{I} \to \tilde{\mathcal{A}}$ by $\tilde{\psi}(\bar{a}) = \psi(a)$, which turns out to be an algebra isomorphism satisfying $\tilde{\psi} \circ \phi^\sigma = \tau^\sigma$, $\sigma = \pm$. $\square$

1.6. Corollary. *Let A be a right faithful associative pair, and $(\mathcal{A}, \varphi)$ be a gr-envelope of A. Then the following are equivalent*:

(i) $(\mathcal{A}, \varphi)$ *is tight on* A,
(ii) $\mathcal{A}$ *is right faithful*,
(iii) $(\mathcal{A}, \varphi)$ *is isomorphic to the standard envelope of* A.

Proof. Apply Proposition 1.5 together with the obvious fact that the set of total right zero divisors of an algebra is an ideal. □

1.7. Notation. To simplify notation, from now on, when dealing with a subpair A of $(\mathcal{A}, \varphi)$ we will assume that $A^\sigma \subseteq \mathcal{A}$, the maps φ^σ will be simply the inclusion maps, and will write $\mathcal{A}$ instead of $(\mathcal{A}, \varphi)$. This will also be applied to the particular case of $(\mathcal{A}, \varphi)$ being an envelope of A.

2. The supersingular left ideal of a superalgebra

The notion of singularity appears naturally in many questions in the theory of modules and rings. In [8], the singular functor of a Grothendieck category is introduced. In particular, for an M in the category R-gr of graded modules over a unital ring R, the graded singular submodule of M is the largest graded submodule contained in $Z(M)$ (the singular submodule of M). Here we study the supersingular ideal of a (not necessarily unital) superalgebra $A = A_0 \oplus A_1$ and relate it to the singular ideals of A_0 (as an algebra) and of A_1 (as an associative triple system).

We will denote by $\mathcal{I}_{\text{gr-l}}(A)$ and $\mathcal{I}^{\text{e}}_{\text{gr-l}}(A)$ the sets of left superideals of A and essential left superideals of A, respectively, while $\mathcal{I}(A)$, $\mathcal{I}_{\text{l}}(A)$ and $\mathcal{I}^{\text{e}}_{\text{l}}(A)$ will stand for the sets of two-sided ideals, left ideals and essential left ideals of A. Throughout this section we will assume that $\sigma, \tau, \alpha, \rho \in \{0, 1\}$. First, some preliminary results.

2.1. Lemma. *Let A be a superalgebra. Let $I \in \mathcal{I}_{\text{gr-l}}(A)$ and $K \in \mathcal{I}_{\text{l}}(A)$. The following assertions hold*:

(i) *If $x \in A_0 \cup A_1$ then $lan(x) := \{y \in A\colon\ yx = 0\} \in \mathcal{I}_{\text{gr-l}}(A)$.*
Moreover, if A is right faithful then:

(ii) *$I \in \mathcal{I}^{\text{e}}_{\text{gr-l}}(A)$ if and only if for every $0 \neq x_\sigma \in A_\sigma$ there exists $a_\tau \in A_\tau$ such that $0 \neq a_\tau x_\sigma \in I_{\tau+\sigma}$.*

(iii) *$K \in \mathcal{I}^{\text{e}}_{\text{l}}(A)$ if and only if for every $0 \neq x \in A$ there exists $a \in A$ such that $0 \neq ax \in K$.*

(iv) *$I \in \mathcal{I}^{\text{e}}_{\text{gr-l}}(A)$ if and only if $I \in \mathcal{I}^{\text{e}}_{\text{l}}(A)$.*

Proof. (i) Is immediate.

(ii) Suppose first $I \in \mathcal{I}^{\text{e}}_{\text{gr-l}}(A)$ and take $0 \neq x_\sigma \in A_\sigma$. Since A has no homogeneous total right zero divisors, Ax_σ (being a left superideal) is nonzero. Our hypothesis applies now so we can choose $a \in A$ with $0 \neq ax_\sigma \in I$. If we decompose the latter into its homogeneous components $ax_\sigma = a_0x_\sigma + a_1x_\sigma$, then at least one is nonzero, and as I is graded we find $a_\tau \in A_\tau$ with $0 \neq a_\tau x_\sigma \in I_{\tau+\sigma}$. To prove the converse take $0 \neq J \in \mathcal{I}_{\text{gr-l}}(A)$, then we are able to find $0 \neq y_\sigma \in J_\sigma \subseteq A_\sigma$ and by applying the hypothesis there exists $a_\tau \in A_\tau$ such that $0 \neq a_\tau y_\sigma \in I$. But $a_\tau y_\sigma \in AJ \subseteq J$, that is, $I \cap J \neq 0$. Forgetting about the grading, one may prove in an similar way (iii).

To prove (iv) use the characterizations given in (ii) and (iii) and follow [15, Lemma I.2.8]. □

2.2. Proposition. *Let A be a superalgebra; define*:

$$Z_{\text{gr-l}}(A)_\sigma := \left\{x \in A_\sigma \colon\ lan(x) \in \mathcal{I}^{\mathrm{e}}_{\text{gr-l}}(A)\right\}, \quad \sigma = 0, 1,$$

and

$$Z_{\text{gr-l}}(A) := Z_{\text{gr-l}}(A)_0 \oplus Z_{\text{gr-l}}(A)_1.$$

Then:

(i) $Z_{\text{gr-l}}(A)$ *is a (two-sided) superideal of A.*
(ii) $Z_{\text{gr-l}}(A) = \{x \in A\colon\ Ix = 0$ *for some* $I \in \mathcal{I}^{\mathrm{e}}_{\text{gr-l}}(A)\}$.
(iii) *If A is right faithful, then $Z_{\text{gr-l}}(A)$ is the biggest superideal of A contained in $Z_{\mathrm{l}}(A)$ and they may not coincide.*

Proof. (i) The only nontrivial part is showing that $Z_{\text{gr-l}}(A)$ is a left superideal of A. Take $x_\sigma \in Z_{\text{gr-l}}(A)_\sigma$ and $a_\tau \in A_\tau$, $\sigma, \tau = 0, 1$. We will show $lan(a_\tau x_\sigma) \in \mathcal{I}^{\mathrm{e}}_{\text{gr-l}}(A)$. We already know that it is a left superideal, and to prove the essentiality we consider $0 \neq J \in \mathcal{I}^{\mathrm{e}}_{\text{gr-l}}(A)$. Pick $0 \neq y_\rho \in J_\rho$. If $y_\rho a_\tau x_\sigma = 0$ then $0 \neq y_\rho \in lan(a_\tau x_\sigma) \cap J_\rho$. In case $y_\rho a_\tau x_\sigma \neq 0$, then a_τ being an homogeneous element easily implies that $Ja_\tau \neq 0$ is a left superideal, and $lan(x_\sigma)$ being essential as a left superideal implies $Ja_\tau \cap lan(x_\sigma) \neq 0$. We may therefore take $z \in J$ with $0 \neq za_\tau \in lan(x_\sigma)$, thus $0 \neq z \in J \cap lan(a_\tau x_\sigma)$ again.

(ii) Consider first $x = x_0 + x_1$ such that $lan(x_\sigma) \in \mathcal{I}^{\mathrm{e}}_{\text{gr-l}}(A)$. Now $I := lan(x_0) \cap lan(x_1) \in \mathcal{I}^{\mathrm{e}}_{\text{gr-l}}(A)$. A straightforward calculation shows that $Ix = 0$. On the other hand, suppose we have $x \in A$ and $I \in \mathcal{I}^{\mathrm{e}}_{\text{gr-l}}(A)$ with $Ix = 0$. Let us see that indeed $Ix_\sigma = 0$, for every $\sigma = 0, 1$. If that is not the case, we have $y \in I$ with $yx_\sigma \neq 0$. But x_σ being homogeneous and I being graded imply that there exists $y_\tau \in I$ with $y_\tau x_\sigma \neq 0$, and consequently $0 \neq y_\tau x \in Ix = 0$, which is absurd. Thus, we have proved $I \subseteq lan(x_\sigma)$; the former being essential and the latter being left graded imply $lan(x_\sigma) \in \mathcal{I}^{\mathrm{e}}_{\text{gr-l}}(A)$, as needed.

(iii) By reasoning as in (ii) we obtain that $Z_{\mathrm{l}}(A) = \{x \in A\colon\ Ix = 0$ for some $I \in \mathcal{I}^{\mathrm{e}}_{\mathrm{l}}(A)\} = \{x \in A\colon\ lan(x) \in \mathcal{I}^{\mathrm{e}}_{\mathrm{l}}(A)\}$ (see [9, p. 30] for the unital case). Take this into account jointly with (ii) and Lemma 2.1(iv) to prove that $Z_{\text{gr-l}}(A)$ is the biggest superideal of A contained in $Z_{\mathrm{l}}(A)$.

To see that $Z_{\text{gr-l}}(A)$ and $Z_{\mathrm{l}}(A)$ do not coincide in general, let F be a field of characteristic 2 and consider the algebra of dual numbers over F:

$$A = F(\varepsilon) = F \cdot 1 \oplus F \cdot \varepsilon \quad \text{with } \varepsilon^2 = 0.$$

It is an easy calculation (note that $(1 + \varepsilon)^2 = 1 + 2\varepsilon + \varepsilon^2 = 1$, since F has characteristic 2) to show that A is a commutative unital superalgebra with a (non-standard) grading given by

$$A_0 = F \cdot 1, \qquad A_1 = \{a \cdot 1 + a \cdot \varepsilon\colon\ a \in F\}.$$

The only nontrivial ideal of A is

$$I = F \cdot \varepsilon;$$

for if $J \in \mathcal{I}(A)$, $J \neq A$, then for every $0 \neq a \in F$ and $b \in F$ we have $a \cdot 1 + b \cdot \varepsilon \notin J$. Otherwise

$$\left(a^{-1} \cdot 1 - ba^{-2} \cdot \varepsilon\right)(a \cdot 1 + b \cdot \varepsilon) = 1 \cdot 1 \in J$$

would lead to $J = A$. So $J \subseteq I$. It is obvious that although I is an ideal, it is not graded: if we consider the element ε, then its homogeneous components ($\varepsilon_0 = -1$ and $\varepsilon_1 = 1 + \varepsilon$) no longer belong to I. Therefore, A is supersimple. Now, since A is unital, then $1 \notin Z_{\mathrm{l}}(A) \cup Z_{\mathrm{gr\text{-}l}}(A)$. On the other hand we have

$$lan(\varepsilon) = F \cdot \varepsilon \in \mathcal{I}_{\mathrm{l}}^{\mathrm{e}}(A) \quad \text{implies } 0 \neq \varepsilon \in Z_{\mathrm{l}}(A).$$

Then it easily follows $Z_{\mathrm{gr\text{-}l}}(A) = 0 \neq I = Z_{\mathrm{l}}(A)$. □

2.3. Definitions. The ideal $Z_{\mathrm{gr\text{-}l}}(A)$ in the proposition above is called the **left supersingular ideal** of A. In a similar way we could talk about the **right supersingular ideal** of A (denoted by $Z_{\mathrm{gr\text{-}r}}(A)$). The **supersingular two-sided ideal** of A is defined as $Z_{\mathrm{gr}}(A) = Z_{\mathrm{gr\text{-}l}}(A) \cap Z_{\mathrm{gr\text{-}r}}(A)$.

This graded definition of singular ideal is consistent with the nongraded one in the sense that whenever we consider a superalgebra A with trivial grading, we obtain $Z_{\mathrm{gr\text{-}l}}(A) = Z_{\mathrm{l}}(A)$.

2.4. Definitions. Let A be a superalgebra. We say A is **left supersingular** if $Z_{\mathrm{gr\text{-}l}}(A) = A$, and we say that A is **left supernonsingular** if $Z_{\mathrm{gr\text{-}l}}(A) = 0$.

To recall the notion of weak left quotient superalgebra, which appears in the statement of the following result, see Section 3.

2.5. Lemma. *Let A be a nonzero left supernonsingular superalgebra and let I be a left superideal of A. Then*:

(i) *A is right faithful.*
(ii) *$I \in \mathcal{I}_{\mathrm{gr\text{-}l}}^{\mathrm{e}}(A)$ if and only if A is a weak left quotient superalgebra of I.*

Proof. (i) If $x_\sigma \in A_\sigma$ is a total (homogeneous) right zero divisor, then $lan(x_\sigma) = A$ implies $x_\sigma \in Z_{\mathrm{gr\text{-}l}}(A)_\sigma = 0$.

(ii) Suppose $I \in \mathcal{I}_{\mathrm{gr\text{-}l}}^{\mathrm{e}}(A)$. If $0 \neq x_\sigma \in A_\sigma$ then $Ix_\sigma \neq 0$ (otherwise $I \subseteq lan(x_\sigma)$ would imply $x_\sigma \in Z_{\mathrm{gr\text{-}l}}(A) = 0$, a contradiction). Take $y_\tau \in I$ such that $y_\tau x_\sigma \neq 0$. By (i), $Ay_\tau x_\sigma$ is a nonzero left superideal of A and by the essentiality of I, $0 \neq a_\alpha y_\tau x_\sigma \in I_{\alpha+\tau+\sigma}$ for some $a_\alpha \in A_\alpha$ (notice that $a_\alpha y_\tau \in I$). For the converse, apply (i) and Lemma 2.1(ii). □

2.6. Remark. The previous lemma still holds if we consider algebras, left ideals and the notions of left singularity, right faithfulness and weak left quotient algebra instead of the analogous graded ones.

2.7. Remark. Note that if A is right faithful, then left nonsingularity implies left supernonsingularity while the converse is not true: see the example in Proposition 2.2(iii). Moreover, such an A is an example of an algebra which is neither nonsingular nor singular, and by considering trivial supergradings we deduce as well that the notions of being supernonsingular and supersingular are not negations of each other.

We are interested in relating the different types of singular ideals we can consider in the different structures we are dealing with, namely: superalgebras, associative pairs and associative triple systems.

2.8. Let A be an associative pair and let $X \subseteq A^\sigma$, $\sigma = \pm$. The **left annihilator of X in A** is defined to be the set

$$lan(X) = lan_A(X) := \{b \in A^{-\sigma}\colon\ bXA^{-\sigma} = 0,\ A^\sigma bX = 0\}.$$

It can be shown [7, (1.2)] that if A is a right faithful associative pair then

$$lan_A(X) = \{b \in A^{-\sigma}\colon\ A^\sigma bX = 0\}.$$

2.9. For an associative triple system T and a subset $X \subset T$, the **left annihilator of X in T** is defined as

$$lan_T(X) := lan_{V(T)}(X),$$

the latter being equal to $lan_{V(T)}(X^\sigma)$, $\sigma = \pm$.

2.10. For a right faithful associative pair A, if we define

$$Z_l(A)^\sigma = \{z \in A^\sigma\colon\ lan_A(z) \in \mathcal{I}_l^e(A)\}, \quad \sigma = \pm,$$

then it turns out that

$$Z_l(A) := \big(Z_l(A)^+, Z_l(A)^-\big)$$

is an ideal of A [7, (1.6)], called the **left singular ideal** of the associative pair A.

2.11. Definition. For an associative triple system T we can define the **left (triple) singular ideal** as

$$Z_l(T) := Z_l\big(V(T)\big)^\sigma, \quad \sigma = \pm.$$

If $A = A_0 \oplus A_1$ is a superalgebra, then the supernonsingularity of A is in fact related with that of A_0 and A_1. In this regard, we have the following results.

2.12. Proposition. *Let A be a right faithful superalgebra such that $A_0 = A_1A_1$. Then*:

(i) *If $I \in \mathcal{I}_{\text{gr-l}}(A)$ then $I_0 = 0$ if and only if $I_1 = 0$.*
(ii) $Z_{\text{gr-l}}(A)_\sigma = Z_{\text{l}}(A_\sigma)$, $\sigma = 0, 1$.

Proof. (i) If $I_0 = 0$ and we take $0 \neq y_1 \in I_1$, then by Lemma 0.9, $0 \neq A_1A_1y_1 \subseteq A_1I_0 = 0$, a contradiction. Conversely, if $I_1 = 0$ and we consider $0 \neq y_0 \in I_0$, then Lemma 0.9 implies $0 \neq A_0y_0 = A_1A_1y_0 \subseteq A_1I_1 = 0$, a contradiction again.

(ii) Consider first $\sigma = 1$ and $0 \neq a_1 \in Z_{\text{l}}(A_1)$. Take $0 \neq L = L_0 \oplus L_1 \in \mathcal{I}_{\text{gr-l}}(A)$. By (i) $L_1 \neq 0 \neq L_0$. Since L_1 is a left ideal of A_1, our hypothesis gives us some $0 \neq l_1 \in L_1 \cap lan_{A_1}(a_1)$, that is, $A_1l_1a_1 = 0$. On the other hand, by Lemma 0.9 A_1 is right faithful, so we find $b_1 \in A_1$ such that $0 \neq b_1l_1 \in L \cap lan_A(a_1)$. To see the other containment, consider $0 \neq a_1 \in A_1$ such that $lan_A(a_1) \in \mathcal{I}^{\text{e}}_{\text{gr-l}}(A)$. If we take $0 \neq J \in \mathcal{I}_{\text{l}}(A_1)$, applying that A is right faithful we can find $0 \neq y_1 \in J$ with $0 \neq Ay_1 \in \mathcal{I}_{\text{gr-l}}(A)$. So $Ay_1 \cap lan_A(a_1) \neq 0$ and by (i) there exists $b_1 \in A_1$ satisfying $0 \neq b_1y_1 \in lan_A(a_1)$. Since $A_0 = A_1A_1$ is right faithful by Lemma 0.9, we find $d_1 \in A_1$ such that $0 \neq d_1b_1y_1 \in J \cap lan_{A_1}(a_1)$.

For the $\sigma = 0$ case we start by taking $0 \neq a_0 \in A_0$ such that $lan_{A_0}(a_0) \in \mathcal{I}^{\text{e}}_{\text{l}}(A_0)$. If we consider $0 \neq K = K_0 \oplus K_1 \in \mathcal{I}_{\text{gr-l}}(A)$, by (i) $K_0 \neq 0$, and since it is a left ideal of A_0 we can find $0 \neq k_0 \in K_0$ such that $k_0 \in lan_{A_0}(a_0) \subseteq lan_A(a_0)$. To prove the other containment we consider $0 \neq a_0 \in A_0$ with $lan_A(a_0) \in \mathcal{I}^{\text{e}}_{\text{gr-l}}(A)$. Take $0 \neq J_0 \in \mathcal{I}_{\text{l}}(A_0)$, and again $0 \neq Ay_0 \in \mathcal{I}_{\text{l}}(A)$ for some $y_0 \in J_0$. Since $Ay_0 \cap lan_A(a_0) \neq 0$, applying (i) we can find $b_0 \in A_0$ such that $0 \neq b_0y_0 \in J_0 \cap lan_{A_0}(a_0)$. $\square$

2.13. Corollary. *For a right faithful superalgebra A with $A_0 = A_1A_1$ the following conditions are equivalent*:

(i) *A is left supernonsingular* (*as a superalgebra*).
(ii) *A_0 is left nonsingular* (*as an algebra*).
(iii) *A_1 is left nonsingular* (*as a triple*).

2.14. Remark. A_0 left nonsingular does not imply A left nonsingular (the superalgebra A considered in Proposition 2.2(iii) satisfies $A_0 = A_1A_1$, A_0 is left nonsingular and A itself is not).

3. Systems of left quotients

Let $A = A_0 \oplus A_1$ be a subsuperalgebra of a superalgebra $B = B_0 \oplus B_1$. In this section we will study when B being a gr-left quotient algebra of A is equivalent to B_0 and B_1 being a left quotient algebra and a left quotient triple system of A_0 and A_1, respectively. See [7] for results on left quotient pairs.

3.1. The notion of left quotient ring was introduced by Utumi in [18]. Let A be a subalgebra of an algebra Q. We say that Q is a (general) **left quotient algebra** of A if for every $p, q \in Q$, with $p \neq 0$, there is an $a \in A$ such that $ap \neq 0$ and $aq \in A$. Notice that an algebra is a left quotient algebra of itself if and only if it is right faithful. In this case then by [18] it has a unique maximal left quotient algebra, which is unital, called the **maximal left quotient algebra** of A. This algebra will be denoted by $Q^{\mathrm{l}}_{\max}(A)$. If we put $p = q$ in the definition of left quotient algebra we speak about a **weak left quotient algebra** of A.

3.2. In [7] a notion of left quotient pair is introduced. Let $A = (A^+, A^-)$ be a subpair of an associative pair $Q = (Q^+, Q^-)$. We say that Q is a **left quotient pair** of A if given $p, q \in Q^\sigma$ with $p \neq 0$ (and $\sigma = +$ or $\sigma = -$) there exist $a \in A^\sigma$, $b \in A^{-\sigma}$ such that

$$abp \neq 0 \quad \text{and} \quad abq \in A^\sigma.$$

Every right faithful associative pair is a left quotient pair of itself.

The notion of left quotient pair extends that of Utumi [18] of left quotient ring since given a subalgebra A of an algebra Q, Q is a left quotient algebra of A if and only if $V(Q)$ is a left quotient pair of $V(A)$.

3.3. Definition. Let S be a subsystem of an associative triple system T. We say that T is a **left quotient triple system** of S if given $p, q \in T$, with $p \neq 0$, there exist $a, b \in S$ such that $abp \neq 0$ and $abq \in S$, equivalently, if $V(T)$ is a left quotient pair of $V(S)$.

3.4. Let $A = \bigoplus_{\sigma \in G} A_\sigma$ be a gr-subalgebra of a gr-algebra $Q = \bigoplus_{\sigma \in G} Q_\sigma$. We will say that Q is a **gr-left quotient algebra** of A if ${}_AA$ is a gr-dense submodule of ${}_AQ$. If given a nonzero element $q_\sigma \in Q_\sigma$ there exists $x_\tau \in A_\tau$ such that $0 \neq x_\tau q_\sigma \in A_{\tau\sigma}$, we say that Q is a **weak gr-left quotient algebra** of A. When $G = \mathbb{Z}/2\mathbb{Z}$ we will speak about a **left quotient superalgebra** and a **weak left quotient superalgebra**. Notice that a gr-algebra is a gr-left quotient algebra of itself if and only if it is gr-right faithful. In this case (see [4]) it has a unique gr-maximal left quotient algebra, which is unital, called the **gr-maximal left quotient algebra** of A. This algebra will be denoted by $Q^{\mathrm{l}}_{\text{gr-max}}(A)$.

3.5. Definitions. Let A be a subsuperalgebra of a superalgebra B. For every $q_i \in B_i$, with $i = 0, 1$, define $(A : q_i) = \{a \in A\colon\ aq_i \in A\}$. We will say that A is **weak right faithful in** B if for every $q_0 \in B_0$, $\operatorname{ran}_{B_1}(A : q_0) = 0$. We will say that A is **right faithful in** B if for every $q_i \in B_i$, $\operatorname{ran}_{B_{i-1}}(A : q_i) = 0$ for each $i \in \{0, 1\}$. This definition has been motivated by the following fact: when $B = A$, the previous condition means A right faithful. Hence every right faithful superalgebra A is right faithful in itself.

3.6. Proposition. *Let A be a subsuperalgebra of a superalgebra B and suppose $A_0 = A_1A_1$.*

(i) *If B is a left quotient superalgebra of A, then B_0 is a left quotient algebra of A_0 and B_1 is a left quotient triple system of A_1.*
(ii) *If B_0 is a left quotient algebra of A_0, B_1 is a left quotient triple system of A_1 and A is weak right faithful in B, then B is a left quotient superalgebra of A and, consequently, a left quotient algebra of A.*

Proof. (i) The fact of B_0 being a left quotient algebra of A_0 was proved in [4, (3.2)]. To see that B_1 is a left quotient triple system of A_1, consider $p_1, q_1 \in B_1$, with $p_1 \neq 0$. Since

B is a left quotient superalgebra of $A = A_0 + A_1$ and $A_0 = A_1A_1$, $0 \neq t_1p_1$ for some $t_1 \in A_1$. Apply that B_0 is a left quotient algebra of A_0 to find $a_0 \in A_0$ such that $a_0t_1p_1 \neq 0$ and $a_0t_1p_1, a_0t_1q_1 \in A_0$. By Lemma 0.9, A_0 has no total right zero divisors, hence $0 \neq A_0a_0t_1p_1 = A_1A_1a_0t_1p_1$. Choose $b_1 \in A_1$ satisfying $0 \neq b_1a_0t_1p_1$. Then $u_1 = b_1a_0 \in A_1$ and t_1 verify: $u_1t_1p_1 \neq 0$ and $u_1t_1q_1 \in A_1$. This shows our claim.

(ii) Consider $p_0, q_0 \in B_0$, with $p_0 \neq 0$. Since B_0 is a left quotient algebra of A_0, there exists $a_0 \in A_0$ such that $a_0p_0 \neq 0$ and $a_0q_0 \in A_0$. Now, consider $0 \neq p_1 \in B_1, q_0 \in B_0$. Apply $0 \neq (A : q_0)p_1$ to find $a_j \in A_j$ satisfying $0 \neq a_jp_1$ and $a_jq_0 \in A_j$. For the third case, take $0 \neq p_0 \in B_0, q_1 \in B_1$. Since B_0 is a left quotient algebra of A_0, $0 \neq A_0p_0 = A_1A_1p_0$, so that $0 \neq t_1p_0$ for some $t_1 \in A_1$. Apply the previous case to find $a_j \in A_j$ satisfying $0 \neq a_jt_1p_0$ and $a_jt_1q_1 \in A_j$. Then $u = a_jt_1$ is an homogeneous element of A such that $0 \neq up_0$ and $uq_1 \in A_0 \cup A_1$. Finally, given $p_1, q_1 \in B_1$, with $p_1 \neq 0$, apply that B_1 is a left quotient triple system of A_1 to find $a_1, b_1 \in A_1$ such that $a_1b_1p_1 \neq 0$ and $a_1b_1q_1 \in A_1$. Then $u_0 = a_1b_1 \in A_0$ satisfies $0 \neq u_0p_1$ and $u_0q_1 \in A_1$. $\square$

3.7. Remark. By Remark 0.10, Lemma 3.6(i) may fail if $A_0 \neq A_1A_1$.

Other examples of right faithful subsuperalgebras in overalgebras (different from the given in Definitions 3.5) can be found in the following result.

3.8. Lemma. *Let B be an oversuperalgebra of a superalgebra A satisfying $A_0 = A_1A_1$ and suppose that B_0 is a left quotient algebra of A_0 and that B_1 is a left quotient triple system of A_1.*

- (i) *If A is left faithful then A is right faithful in B.*
- (ii) *If A is left supernonsingular* (*in particular, if it is left nonsingular*) *then B is a left quotient algebra of A and A is right faithful in B. Moreover,*
- (i)′ *The left faithfulness of A in* (i) *can be replaced by the left faithfulness of A_i for $i = 0$ or $i = 1$.*
- (ii)′ *The left supernonsingularity of A in* (ii) *can be replaced by the left nonsingularity of A_i for $i = 0$ or $i = 1$.*

Proof. (i) We will prove the case $i = 0$. The other one is similar. Suppose $0 \neq b_1 \in \mathrm{ran}_{B_1}(A : q_0)$ for some $q_0 \in B_0$. Apply that B_1 is a left quotient triple system of A_1 to find $u_1, v_1 \in A_1$ such that $0 \neq u_1v_1b_1 \in A_1$. Since A is left faithful and $A_0 = A_1A_1$, there exists $w_1 \in A_1$ such that $u_1v_1b_1w_1 \neq 0$. B_0 being a left quotient algebra of A_0 implies $a_0u_1v_1b_1w_1 \neq 0$ and $a_0u_1v_1q_0 \in A_0$ for some $a_0 \in A_0$. Now, $a_0u_1v_1 \in (A : q_0)$ and $b_1 \in \mathrm{ran}_{B_1}(A : q_0)$ imply $a_0u_1v_1b_1 = 0$, a contradiction.

(ii) We prove first that B is a left quotient algebra of A. Given $p_0, q_0 \in B_0$, with $p_0 \neq 0$, apply that B_0 is a left quotient algebra of A_0 to find $a_0 \in A_0$ such that $a_0p_0 \neq 0$ and $a_0q_0 \in A_0$. If $p_1, q_1 \in B_1$, with $p_1 \neq 0$, by using that B_1 is a left quotient triple system of A_1 we find $u_1, v_1 \in A_1$ satisfying $0 \neq u_1v_1p_1$ and $u_1v_1q_1 \in A_1$. Now, consider $0 \neq p_0 \in B_0$ and $q_1 \in B_1$; apply that B_0 is a left quotient algebra of A_0 to find $a_0 \in A_0$ such that $0 \neq a_0p_0 \in A_0$. Since A is right faithful (by Lemma 2.5) and $A_0 = A_1A_1$, $b_1a_0p_0 \neq 0$ for some $b_1 \in A_1$. Notice that $V(B_1)$ is a left quotient pair of $V(A_1)$ and that $V(A_1)$ is left nonsingular (by Lemma 2.5(i) and Corollary 2.13); by [7, (2.4)] $(A_1 : b_1a_0q_1)b_1a_0p_0 \neq 0$, hence there exists $c_1 \in A_1$ satisfying $c_1b_1a_0p_0 \neq 0$ and $c_1b_1a_0q_1 \in A_1$. The element $u_0 = c_1b_1a_0 \in A_0$ satisfies: $u_0p_0 \neq 0$ and $u_0q_1 \in A_1$.

Finally, given $0 \neq p_1 \in B_1$ and $q_0 \in B_0$, apply that B_1 is a left quotient triple system of A_1 to find $a_1 \in A_1$ such that $a_1p_1 \neq 0$. By the previous case there exists $u_0 \in A_0$ satisfying $0 \neq u_0a_1p_1$ and $u_0a_1q_0 \in A_1$.

The equality $\mathrm{ran}_{B_{1-i}}(A : q_i) = 0$ for every $q_i \in B_i$ and every $i = 0, 1$ follows from the fact of B being a left quotient superalgebra of A.

(i)$'$ Under the conditions of the main statement, A is left faithful if and only if A_0 and A_1 are left faithful (by Lemma 0.9). Suppose A_0 left faithful, and consider $a_1 \in A_1$ such that $a_1A_1A_1 = 0$. If $a_1 \neq 0$, $A_1a_1 \neq 0$ by the right faithfulness of A_1. Apply that A_0 is left faithful to have $0 \neq A_1a_1A_0 = A_1a_1A_1A_1$, which is a contradiction.

Now, suppose A_1 left faithful, and consider $a_0 \in A_0$ satisfying $a_0A_0 = 0$. Then $a_0A_1A_1A_1 = a_0A_0A_1 = 0$. Since A_1 has no total right zero divisors, $a_0A_1 = 0$. If $a_0 \neq 0$, apply the right faithfulness of A_0 to have $0 \neq A_0a_0 = A_1A_1a_0$. Apply again the left faithfulness of A_1 to obtain $0 \neq A_1a_0A_1A_1$, a contradiction.

(ii)$'$ follows by Lemma 0.9 and Corollary 2.13. $\square$

3.9. Remark. The converses of (i) and (ii) in the previous lemma are not true. Consider $A = B$ and take into account that right faithfulness implies neither left faithfulness nor left supernonsingularity.

3.10. Corollary. *Let A be a right faithful subsuperalgebra of a superalgebra B and suppose $A_0 = A_1A_1$. If A is left faithful (equivalently gr-left faithful) or gr-left nonsingular, then B is a left quotient superalgebra of A if and only if B_0 is a left quotient algebra of A_0 and B_1 is a left quotient triple system of A_1.*

Proof. Apply Lemmas 3.6(i) and 3.8. $\square$

4. The maximal left quotient system of an associative system

4.1. Let A be an associative pair and denote by $(\mathcal{E}, e)$ and $\mathcal{A}$ its standard imbedding and standard envelope, respectively. Then $\mathcal{A}$ and $\mathcal{E}$ can be considered as superalgebras by defining

$$\mathcal{A}_0 := \mathcal{A}_{12}\mathcal{A}_{21} \oplus \mathcal{A}_{21}\mathcal{A}_{12}, \qquad \mathcal{A}_1 := \mathcal{A}_{12} \oplus \mathcal{A}_{21}$$

and

$$\mathcal{E}_0 := e\mathcal{E}e \oplus (1-e)\mathcal{E}(1-e), \qquad \mathcal{E}_1 := \mathcal{A}_1.$$

Moreover, by 1.2, $\mathcal{A}_0 = \mathcal{A}_1\mathcal{A}_1$, although the same is not true, in general, for $\mathcal{E}_0$. When $\mathcal{E}_0 = \mathcal{E}_1\mathcal{E}_1$, then $\mathcal{E} = \mathcal{A}$ and A is said to be a **unital associative pair**. As it is not difficult to see, the pair A is unital if and only if e is a full idempotent in $\mathcal{E}$, if and only if $\mathcal{A} = \mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1$ is a strongly graded superalgebra.

4.2. Remark. Notice that the standard envelope of an associative pair A is not necessarily a strongly graded superalgebra. For a commutative ring R, take

$$\mathcal{A}=\begin{pmatrix} \langle x^2\rangle & \langle x\rangle \\ \langle x\rangle & \langle x^2\rangle \end{pmatrix},$$

where $\langle f(x)\rangle$ denotes the ideal generated by $\{f(x)\}$ in the polynomial ring $R[x]$. Then the standard envelope of the associative pair $A=V(\langle x\rangle)$ is isomorphic to $\mathcal{A}$ (consider A as a subpair of $(\mathcal{A},\varphi)$, where $\varphi=(\varphi^+,\varphi^-)$ is given by

$$\begin{aligned} \varphi^+ : A^+ &\to \mathcal{A} & \qquad \varphi^- : A^- &\to \mathcal{A}, \\ a^+ &\mapsto \begin{pmatrix} 0 & a^+ \\ 0 & 0 \end{pmatrix} & a^- &\mapsto \begin{pmatrix} 0 & 0 \\ a^- & 0 \end{pmatrix} \end{aligned}$$

and apply Corollary 1.6). Moreover,

$$\mathcal{A}_0\mathcal{A}_1=\begin{pmatrix} \langle x^2\rangle & 0 \\ 0 & \langle x^2\rangle \end{pmatrix}\begin{pmatrix} 0 & \langle x\rangle \\ \langle x\rangle & 0 \end{pmatrix}=\begin{pmatrix} 0 & \langle x^3\rangle \\ \langle x^3\rangle & 0 \end{pmatrix}\neq \mathcal{A}_1.$$

4.3. Lemma. *Let B be a left quotient pair of an associative pair A, and denote by $\mathcal{A}$ and $\mathcal{B}$ their standard envelopes. Then*:

(i) $\mathcal{A}_{ji}b_{ii}\neq 0$ *for every* $0\neq b_{ii}\in\mathcal{B}_{ii}$, $i,j\in\{1,2\}$.
(ii) $\mathcal{B}_{ii}$ *is a left quotient algebra of* $\mathcal{A}_{ii}$ *for* $i\in\{1,2\}$.

Proof. Notice that by [7, (2.5)(i)], $\mathcal{A}\subseteq\mathcal{B}$.

(i) The case $i\neq j$ is [7, (2.6)]. Now, suppose $i=j$. By the previous case, $a_{ki}b_{ii}\neq 0$ for some $a_{ki}\in\mathcal{A}_{ki}$, with $k\neq i$ and $k,i\in\{1,2\}$. Apply that B is a left quotient pair of A to find $(x_{ik},x_{ki})\in(\mathcal{A}_{ik},\mathcal{A}_{ki})$ such that $0\neq x_{ki}x_{ik}a_{ki}b_{ii}\in\mathcal{A}_{ki}\mathcal{A}_{ii}b_{ii}$. This shows $\mathcal{A}_{ii}b_{ii}\neq 0$.

(ii) Consider $b_{ii},c_{ii}\in\mathcal{B}_{ii}$, with $b_{ii}\neq 0$. By (i) there exists $a_{ji}\in\mathcal{A}_{ji}$, with $j\neq i$ and $j\in\{1,2\}$, such that $a_{ji}b_{ii}\neq 0$. Apply that B is a left quotient pair of A and take $(x_{ij},x_{ji})\in(\mathcal{A}_{ij},\mathcal{A}_{ji})$ satisfying $x_{ji}x_{ij}a_{ji}b_{ii}\neq 0$ and $x_{ji}x_{ij}a_{ji}c_{ii}\in\mathcal{A}_{ji}$. Since A is right faithful, $y_{ij}x_{ji}x_{ij}a_{ji}b_{ii}\neq 0$ for some $y_{ij}\in\mathcal{A}_{ij}$. Then $u_{ii}=y_{ij}x_{ji}x_{ij}a_{ji}\in\mathcal{A}_{ii}$ satisfies $u_{ii}b_{ii}\neq 0$ and $u_{ii}c_{ii}\in\mathcal{A}_{ii}$. $\square$

4.4. Corollary. *Let A be a right faithful associative pair and denote by $\mathcal{A}$ and $(\mathcal{E},e)$ its standard envelope and standard imbedding, respectively. Then $e\mathcal{E}e$ is a left quotient algebra of $e\mathcal{A}e$.*

Proof. We first show $\mathrm{ran}_{\mathcal{A}}(e\mathcal{A})=0$. Suppose $0\neq x\in\mathrm{ran}_{\mathcal{A}}(e\mathcal{A})$. If $x_{11}\neq 0$ then, by Lemma 4.3(i), $0\neq\mathcal{A}_{11}x_{11}=\mathcal{A}_{12}\mathcal{A}_{21}x_{11}\subseteq e\mathcal{A}xe=0$, a contradiction. If $x_{12}\neq 0$ then (since A has no total right zero divisors) $0\neq\mathcal{A}_{12}\mathcal{A}_{21}x_{12}\subseteq e\mathcal{A}x(1-e)=0$, a contradiction. Analogously we obtain $x_{22}=x_{21}=0$ and hence $x=0$. Now, the result follows by [7, (1.5)] and [3, (1.7)]. $\square$

4.5. Lemma. *Let A be a right faithful associative pair, and denote by $\mathcal{A}$ and $(\mathcal{E}, e)$ its standard envelope and standard imbedding, respectively. Then, for every left quotient algebra $\mathcal{Q}$ of $\mathcal{A}$ such that $\mathcal{Q}e + e\mathcal{Q} + \mathcal{Q}(1-e) + (1-e)\mathcal{Q} \subseteq \mathcal{Q}$ we have that $Q := (e\mathcal{Q}(1-e), (1-e)\mathcal{Q}e)$ is a left quotient pair of A.*

Proof. Notice that the products $u\mathcal{Q}v$, for $u, v \in \{1, e, 1-e\}$ make sense by considering $1, u, v, \mathcal{Q}$ inside $Q^{\mathrm{l}}_{\max}(\mathcal{Q}) =$ (by [18, (1.14)]) $Q^{\mathrm{l}}_{\max}(\mathcal{A}) =$ (by [7, (1.5)(ii)] and [18, (1.14)]) $Q^{\mathrm{l}}_{\max}(\mathcal{E})$.

Consider $p_{12}, q_{12} \in e\mathcal{Q}(1-e)$, with $p_{12} \neq 0$. Since $\mathcal{Q}$ is a left quotient algebra of $\mathcal{A}$ there exists $a \in \mathcal{A}$ such that $ap_{12} \neq 0$ and $ap_{12}, aq_{12} \in \mathcal{A}$. Suppose first $a_{11}p_{12} \neq 0$. Then $a_{11}p_{12}, a_{11}q_{12} \in e\mathcal{A}(1-e)$. Apply that A is a left quotient pair of A to find $x_{12}, x_{21} \in A$ satisfying $x_{12}x_{21}a_{11}p_{12} \neq 0$, $x_{12}x_{21}a_{11}q_{12} \in \mathcal{A}_{12}$. Notice that $x_{21}a_{11} \in \mathcal{A}_{21}$. Now, suppose $a_{21}p_{12} \neq 0$. Since $\mathcal{A}$ has no total right zero divisors, $0 \neq \mathcal{A}a_{21}p_{12} \subseteq \mathcal{A}_{12}a_{21}p_{12} + \mathcal{A}_{22}a_{21}p_{12} = \mathcal{A}_{12}a_{21}p_{12} + \mathcal{A}_{21}\mathcal{A}_{12}a_{21}p_{12}$; hence $b_{12}a_{21}p_{12} \neq 0$ for some $b_{12} \in \mathcal{A}_{12}$. The element $c_{11} = b_{12}a_{21} \in \mathcal{A}$ satisfies $c_{11}p_{12} \neq 0$, $c_{11}p_{12}, c_{11}q_{12} \in \mathcal{A}$, and the previous case applies. □

4.6. Remark. [7, (2.5)(ii)] is a particular case of the previous result.

4.7. Lemma. *Let B be a left quotient pair of an associative pair A. Denote by $(\mathcal{B}, e)$ and $(\mathcal{A}, e)$ their standard envelopes and by $\mathcal{Q}^{\mathcal{B}}$ and $\mathcal{Q}^{\mathcal{A}}$ their maximal left quotient algebras. Then $u\mathcal{Q}^{\mathcal{B}}u$ is a left quotient algebra of $u\mathcal{A}u$, for $u \in \{e, 1-e\}$. In particular, $u\mathcal{Q}^{\mathcal{A}}u$ is a left quotient algebra of $u\mathcal{A}u$.*

Proof. We will prove the result for $u = e$. Notice that by [7, (2.5)(i)] the idempotent e is the same for $\mathcal{A}$ and $\mathcal{B}$; moreover, we may consider $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{Q}^{\mathcal{B}}$.

$\mathrm{ran}_{\mathcal{B}e}(e\mathcal{B}) = 0$. Indeed, consider $0 \neq be \in \mathcal{B}e$; by [7, (1.5)] $\mathcal{B}be \neq 0$, so $e\mathcal{B}be \neq 0$ or $(1-e)\mathcal{B}be \neq 0$; in the first case, $be \notin \mathrm{ran}_{\mathcal{B}e}(e\mathcal{B})$; in the second one, choose $c \in \mathcal{B}$ satisfying $0 \neq (1-e)cbe \in B$ and apply that B is a left quotient pair of A to find $(x, y) \in A$ such that $0 \neq yx(1-e)cbe \in \mathcal{A}_{21}\mathcal{B}be = \mathcal{A}_{21}e\mathcal{B}be$; then $be \notin \mathrm{ran}_{\mathcal{B}e}(e\mathcal{B})$.

Now we see $\mathrm{ran}_{\mathcal{B}(1-e)}(e\mathcal{B}) = 0$. Consider $b \in \mathcal{B}$ such that $b(1-e) \neq 0$. By [7, (1.5)], $\mathcal{B}b(1-e) \neq 0$. If $e\mathcal{B}b(1-e) \neq 0$ we have $b(1-e) \notin \mathrm{ran}_{\mathcal{B}(1-e)}(e\mathcal{B})$. If $(1-e)\mathcal{B}b(1-e) \neq 0$, by Lemma 4.3(i) $0 \neq e\mathcal{A}(1-e)\mathcal{B}b(1-e)$ and so $b(1-e) \notin \mathrm{ran}_{\mathcal{B}(1-e)}(e\mathcal{B})$. Since $\mathrm{ran}_{\mathcal{B}}(e\mathcal{B}) = \mathrm{ran}_{\mathcal{B}e}(e\mathcal{B}) \oplus \mathrm{ran}_{\mathcal{B}(1-e)}(e\mathcal{B}) = 0$, we may apply [3, (1.7)] to the algebras $\mathcal{B}$ and $\mathcal{Q}^{\mathcal{B}}$ and to the idempotent e to obtain that $e\mathcal{Q}^{\mathcal{B}}e$ is a left quotient algebra of $e\mathcal{B}e$. If we apply Lemma 4.3(ii) and the transitivity of the relation "being a left quotient algebra of," we finish the proof. □

4.8. Definition. Let A be a subpair of an associative pair $B \subseteq \mathcal{B}$, where $\mathcal{B}$ is the standard envelope of B. We will say that A is **right faithful in** B if

$$\mathrm{ran}_{\mathcal{B}_{12}}\left(\mathcal{A}_{21} : \sum_{i=1}^{m} p_{12}^{i} p_{21}^{i}\right) = 0 \quad \text{and} \quad \mathrm{ran}_{\mathcal{B}_{21}}\left(\mathcal{A}_{12} : \sum_{j=1}^{n} q_{21}^{j} q_{12}^{j}\right) = 0$$

for every finite family $(p_{12}^{i}, p_{21}^{i}), (q_{12}^{j}, q_{21}^{j}) \in B$, with $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$.

4.9. Definition. An associative triple system A is said to be **right faithful** in an associative triple oversystem B when $V(A)$ is a right faithful associative pair in $V(B)$.

4.10. Lemma. *Let A be a subpair of an associative pair $B \subseteq \mathcal{B}$, where $\mathcal{B}$ is the standard envelope of B, and denote by $\mathcal{A}$ the graded algebra generated by A inside $\mathcal{B}$.*

(i) *A is right faithful in B if and only if $\mathcal{A}$ is weak right faithful in $\mathcal{B}$.*
Suppose that B is a left quotient pair of A.
(ii) *$\mathcal{A}$ is the standard envelope of A.*
(iii) *If A is left faithful or left nonsingular then $\mathcal{A}$ is right faithful in $\mathcal{B}$. In particular A is right faithful in B.*

Proof. Consider $(q_{11}, q_{22}) \in (\mathcal{B}_{11}, \mathcal{B}_{22})$, and put $q_0 := q_{11} + q_{22}$. Then

$$\operatorname{ran}_{\mathcal{B}_1}(\mathcal{A} : q_0) = \operatorname{ran}_{\mathcal{B}_{12}}(\mathcal{A}_{21} : q_{11}) \oplus \operatorname{ran}_{\mathcal{B}_{21}}(\mathcal{A}_{12} : q_{22}). \tag{1}$$

Indeed, the containment "$\subseteq$" is not difficult to prove. For the converse, consider $b_{12} \in \operatorname{ran}_{\mathcal{B}_{12}}(\mathcal{A}_{21} : q_{11})$. Since we want to prove $(\mathcal{A} : q_0)b_{12} = 0$, take $a \in (\mathcal{A} : q_0)$. Then $a_{21}, \mathcal{A}_{21}a_{11}$ belong to $(\mathcal{A}_{21} : q_{11})$ and so $a_{21}b_{12} = 0 = \mathcal{A}_{21}a_{11}b_{12}$. Since B is a left quotient pair of A, $a_{11}b_{12} = 0$, which proves our claim. Analogously we obtain $\operatorname{ran}_{\mathcal{B}_{21}}(\mathcal{A}_{12} : q_{22}) \subseteq \operatorname{ran}_{\mathcal{B}_1}(\mathcal{A} : q_0)$.

Now, (i) follows immediately from (1).

(ii) By Corollary 1.6 it is enough to prove that $\mathcal{A}$ is right faithful, equivalently (by [4, (1.10)]) $\mathcal{A}$ is right superfaithful. If $\mathcal{A}a_1 = 0$ for some $a_1 \in \mathcal{A}_1 := A^+ \oplus A^-$, then $\mathcal{A}_1 a_1 = 0$. Since A is right faithful (equivalently $\mathcal{A}_1$ is right faithful), $a_1 = 0$. Suppose now $\mathcal{A}a_0 = 0$ for some $a_0 \in \mathcal{A}_0 = \mathcal{A}_1\mathcal{A}_1$. Since B is right faithful, by [7, (1.5)] $\mathcal{B}$ is right faithful. Hence, a_0 is not a total right zero divisor in $\mathcal{B}$. Apply $\mathcal{B} = \mathcal{B}_0 \oplus \mathcal{B}_1 = \mathcal{B}_1\mathcal{B}_1 \oplus \mathcal{B}_1$ to find $x_1 \in \mathcal{B}_1$ satisfying $x_1 a_0 \neq 0$. Since B is a left quotient pair of A (equivalently $\mathcal{B}_1$ is a left quotient triple system of $\mathcal{A}_1$) there exist $b_1, c_1 \in \mathcal{A}_1$ such that $b_1c_1x_1a_0 \neq 0$ and $b_1c_1x_1 \in \mathcal{A}_1$. But $b_1c_1x_1a_0\mathcal{A}_1a_0 \subseteq \mathcal{A}a_0 = 0$, a contradiction.

(iii) If A is right faithful, and left faithful or left nonsingular, by [7, (1.5), (2.14)], $\mathcal{A}$ is right faithful, and left faithful or left nonsingular. On the other hand, $\mathcal{B}$ is a left quotient algebra of $\mathcal{A}$ (apply [7, (2.5)]). Notice that $\mathcal{A}_0 = \mathcal{A}_1\mathcal{A}_1$. Moreover, $\mathcal{B}_1$ is a left quotient triple system of $\mathcal{A}_1$ (since B is a left quotient pair of A), and $\mathcal{B}_0$ is a left quotient algebra of $\mathcal{A}_0$ (apply Lemma 4.3), which imply, by virtue of Lemma 3.8, $\mathcal{A}$ right faithful in $\mathcal{B}$. Now the result follows by (i). $\square$

4.11. Remark. The converse of Lemma 4.10(iii) is not true, that is, there are examples of associative pairs $A \subseteq B$, with B a left quotient pair of A, and A right faithful in B, and such that A is neither left faithful nor left nonsingular: take $A = B$. Then being A right faithful in A says merely A is right faithful, but right faithfulness implies neither left faithfulness nor left nonsingularity. For the first example, consider a field F and take

$$A = B = \left(\begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix} \right),$$

which is a right but not a left faithful associative pair. For the second one take, for example, $A = (\mathcal{A}, \mathcal{A})$, for $\mathcal{A}$ a right faithful algebra with $Q^{\mathrm{l}}_{\mathrm{max}}(\mathcal{A})$ not being von Neumann regular. □

4.12. Theorem. *Let A be a left quotient subpair of an associative pair B such that A is right faithful in B, and denote by $\mathcal{A}$, $(\mathcal{E}^{\mathcal{A}}, e)$ and $\mathcal{B}$, $(\mathcal{E}^{\mathcal{B}}, e)$ the standard envelopes and standard imbeddings of A and B, respectively. Then*

(i) $\mathcal{Q} := Q^{\mathrm{l}}_{\text{gr-max}}(\mathcal{A}) = Q^{\mathrm{l}}_{\mathrm{max}}(\mathcal{A}) = Q^{\mathrm{l}}_{\mathrm{max}}(\mathcal{B}) = Q^{\mathrm{l}}_{\text{gr-max}}(\mathcal{B})$.
(ii) $Q := (e\mathcal{Q}(1-e), (1-e)\mathcal{Q}e)$ *is a left quotient pair of A.*

Proof. (i) By Lemma 4.3(ii), $\mathcal{B}_0$ is a left quotient algebra of $\mathcal{A}_0$; since $\mathcal{B}_1$ is a left quotient triple system of $\mathcal{A}_1$ (because B is a left quotient pair of A) and $\mathcal{A}$ is right faithful in $\mathcal{B}$ (by Lemma 4.10(i)), we obtain from Lemma 3.6(ii) that $\mathcal{B}$ is a left quotient superalgebra of $\mathcal{A}$ and, consequently, a left quotient algebra of $\mathcal{A}$. Hence, by [18, (1.14)] and [4, (2.8), (1.12)], $\mathcal{Q} := Q^{\mathrm{l}}_{\mathrm{max}}(\mathcal{A}) = Q^{\mathrm{l}}_{\mathrm{max}}(\mathcal{B})$ and $Q^{\mathrm{l}}_{\text{gr-max}}(\mathcal{A}) = Q^{\mathrm{l}}_{\text{gr-max}}(\mathcal{B})$. To finish the proof, apply [4, (1.18), (2.3), (2.4)] and the fact that $\mathcal{Q}$ is graded and contains $\mathcal{A}$ as a gr-subalgebra (notice that the grading is given by the idempotent e).

(ii) is Lemma 4.5. □

4.13. Definition. Given a right faithful associative pair A with standard envelope and imbedding $\mathcal{A}$ and $(\mathcal{E}, e)$, respectively, write $\mathcal{Q} := Q^{\mathrm{l}}_{\mathrm{max}}(\mathcal{A})$. By Lemma 4.5, $Q := (e\mathcal{Q}(1-e), (1-e)\mathcal{Q}e)$ is a left quotient pair of A. Moreover, if B is a left quotient pair of A such that A is right faithful in B, then by Theorem 4.12(i), $\mathcal{Q} = Q^{\mathrm{l}}_{\text{gr-max}}(\mathcal{B})$ and hence there exists a monomorphism (of associative pairs) from B into Q which is the identity when restricted to A. The associative pair Q is called the **maximal left quotient pair** of A and will be denoted by $Q^{\mathrm{l}}_{\mathrm{max}}(A)$. It is maximal among all left quotient pairs of A in which A is right faithful in the sense previously explained.

4.14. Remark. The previous definition strictly generalizes that of [7, (2.11)]. Moreover, it cannot be improved.

Proof. Indeed, when A is an associative pair without total right and total left zero divisors, or it is left nonsingular, the definition coincides with the given in [7, (2.11)] because by Lemma 4.10(ii), under this conditions, A is right faithful in every left quotient pair of A. It strictly generalizes [7, (2.11)] by virtue of Remark 4.11.

For the second sentence, suppose that B is a left quotient pair of A such that there exists a monomorphism (of associative pairs) from B into $Q := (eQ^{\mathrm{l}}_{\mathrm{max}}(\mathcal{A})(1-e), (1-e)Q^{\mathrm{l}}_{\mathrm{max}}(\mathcal{A})e)$ which is the identity when restricted to A. Identify B with its image inside Q and denote by $\mathcal{Q}$ the standard envelope of Q. Then $A \subseteq B \subseteq Q \subseteq \mathcal{Q} \subseteq Q^{\mathrm{l}}_{\mathrm{max}}(\mathcal{A})$ (notice that $\mathcal{Q}$ and $Q^{\mathrm{l}}_{\mathrm{max}}(\mathcal{A})$ may not coincide, see [7, (2.12)] for an example). Then, $\mathcal{Q}$ being a gr-left quotient algebra of $\mathcal{A}$ implies (by [4, (1.17), (1.18)]) that for every $q_0 \in \mathcal{Q}_0$, $\mathcal{A}$ is a left quotient superalgebra of $(\mathcal{A} : q_0)$ and so $\mathcal{Q}$ is a left quotient superalgebra of $(\mathcal{A} : q_0)$. Hence $\mathrm{ran}_{\mathcal{Q}_1}(\mathcal{A} : q_0) = 0$. By Lemma 4.10(i), A is right faithful in Q. Now, denote by $\mathcal{B}$ the graded algebra generated by B inside $\mathcal{Q}$. Then $\mathcal{B}$ is the standard envelope of B: since Q is a left quotient pair of A and $A \subseteq B \subseteq Q$, Q is a left quotient pair of B; this implies, by Lemma 4.10(ii), our statement. Finally, for every finite family $\{(p_{12}^i, p_{21}^i)\} \subseteq (\mathcal{B}_{12}, \mathcal{B}_{21})$, $\mathrm{ran}_{\mathcal{B}_{12}}(\mathcal{A}_{21}\colon \sum_i p_{12}^i p_{21}^i) = \mathrm{ran}_{\mathcal{Q}_{12}}(\mathcal{A}_{21}\colon \sum_i p_{12}^i p_{21}^i) \cap \mathcal{B}_{12} = 0$. This fact and the analogue obtained by exchanging the roles of 1 and 2, complete the proof. □

5. The maximal left quotient system of an associative triple system

5.1. Let A be an associative triple system and denote by $\mathcal{A}$ the standard envelope of $V(A) := (A, A)$. Consider the natural inclusion (τ^+, τ^-), with $\tau^\sigma : V(A)^\sigma \to \mathcal{A}$, for $\sigma = \pm$.

Then the linear map $\tau : \mathcal{A}_{-1} \oplus \mathcal{A}_0 \oplus \mathcal{A}_1 \to \mathcal{A}_{-1} \oplus \mathcal{A}_0 \oplus \mathcal{A}_1$ satisfying

$$\tau\Big(\tau^+(u) + \sum \tau^+(a_i)\tau^-(b_i) + \sum \tau^-(c_i)\tau^+(d_i) + \tau^-(v)\Big)$$
$$:= \tau^-(u) + \sum \tau^-(a_i)\tau^+(b_i) + \sum \tau^+(c_i)\tau^-(d_i) + \tau^+(v)$$

for every $u, v, a_i, b_i \in A$, is an involutory gr-homomorphism of gr-algebras, i.e., $\tau^2 = 1_{\mathcal{A}}$, $\tau(a_l) \in \mathcal{A}_{-l}$, for $l = -1, 0, 1$, and $\tau(ab) = \tau(a)\tau(b)$.

5.2. Theorem. *Let A be a right faithful associative triple system and let $\mathcal{A}$ and τ be as in* 5.1. *Denote $\mathcal{Q}_{-1} \oplus \mathcal{Q}_0 \oplus \mathcal{Q}_1 = \mathcal{Q} := Q^{\mathrm{l}}_{\text{gr-max}}(\mathcal{A}) = Q^{\mathrm{l}}_{\max}(\mathcal{A})$. Then*:

(i) *τ can be extended to an involutory gr-homomorphism of gr-algebras $\tilde{\tau} : \mathcal{Q}_{-1} \oplus \mathcal{Q}_0 \oplus \mathcal{Q}_1 \to \mathcal{Q}_{-1} \oplus \mathcal{Q}_0 \oplus \mathcal{Q}_1$ which coincides with τ when restricted to $\mathcal{A}$.*

(ii) *$Q := \mathcal{Q}_1$ with the triple product given by*: *$x \cdot y \cdot z := xy^{\tilde{\tau}}z$* (*being the juxtaposition the product in $\mathcal{Q}$ and $y^{\tilde{\tau}}$ the image of y via $\tilde{\tau}$*) *is an associative triple system and a left quotient triple system of A.*

(iii) *Q is maximal among all left quotient triple systems of A in which A is right faithful in the sense that if T is a left quotient triple system of A, then there exists a monomorphism from T into Q* (*of associative triple systems*) *which restricted to A is the identity.*

Proof. (i) It is easy to see that $\mathcal{A}$ is a gr-left quotient algebra of a gr-ideal $\mathcal{I}$ if and only if $\mathcal{A}$ is a gr-left quotient algebra of $\mathcal{I}^\tau$ and that for any $f \in HOM_{\mathcal{A}}(\mathcal{I}^\tau, \mathcal{A})_l$ (where $HOM_{\mathcal{A}}(\mathcal{I}^\tau, \mathcal{A})_l$ denotes the set of all graded homomorphisms of left $\mathcal{A}$-gr-modules from $\mathcal{I}^\tau$ to $\mathcal{A}$) the map $f^\tau : \mathcal{I}^\tau \to \mathcal{A}$ given by $f^\tau(y^\tau) := f(y)^\tau$ lies in $HOM_{\mathcal{A}}(\mathcal{I}^\tau, \mathcal{A})_{-l}$, for $l = -1, 0, 1$. Moreover, $\tilde{\tau} : \mathcal{Q} \to \mathcal{Q}$ defined by $[f, \mathcal{I}]^{\tilde{\tau}} = [f^\tau, \mathcal{I}^\tau]$ satisfies the desired conditions.

(ii) It is immediate to see that Q is a left quotient triple system with the given triple product. Now, let p, q be in Q, with $p \neq 0$. Apply Theorem 4.12(ii) to find $(a, b) \in V(A)$ such that $0 \neq abp = a \cdot b^\tau \cdot p$ and $a \cdot b^\tau \cdot q = abq \in A$. This proves that A is a left quotient triple system of A.

(iii) If B is a left quotient triple system of A in which A is right faithful then, by Definition 3.3, $V(B)$ is a left quotient pair of $V(A)$. Clearly, the right faithfulness of A inside B can be read as the right faithfulness of $V(A)$ inside $V(B)$. By Definition 4.13 $\mathcal{Q} = Q^{\mathrm{l}}_{\text{gr-max}}(\mathcal{E}(V(B))) = Q^{\mathrm{l}}_{\max}(\mathcal{E}(V(B))) = Q^{\mathrm{l}}_{\text{gr-max}}(\mathcal{E}(V(A))) = Q^{\mathrm{l}}_{\max}(\mathcal{E}(V(A)))$, where $\mathcal{E}(V(-))$ denotes the envelope of $V(-)$, and $(\mathcal{Q}_{-1}, \mathcal{Q}_1)$ is a left quotient pair of $V(A)$. By (i) and (ii), B can be seen as a subtriple of Q. $\square$

5.3. Definition. For every associative triple system A the left quotient triple system Q defined in Theorem 5.2 is called the **maximal left quotient triple system of** A.

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